Supplementary Material for "Hybrid architecture for encoded measurement-based quantum computation"

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GRAPH STATES

A graph G is a set of vertices V and edges E. In quantum information one associates a $N = |V|$ qubit graph state $|G\rangle$ with a graph G in the following way [1, 2]: for each vertex j one defines an operator $K_j = X_j \prod_{i\in N(j)} Z_i$, where $N(j)$ denotes the neighborhood of vertex j, i.e., all vertices that are connected to j by an edge, and X and Z are the usual Pauli matrices. The graph state $|G\rangle$ is then uniquely defined as the eigenstate with eigenvalue $+1$ for all operators K_j. Alternatively $|G\rangle$ can be defined as the state resulting from the application a controlled phase gate ($CZ = diag(1, 1, 1, -1)$ in computational basis) between any pairs of qubits, initially in state $|+\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle)$, connected by an edge in the graph. Graph states are important in the context of measurement-based quantum computation and quantum error correction.

QUANTUM GATES

Measurement-based implementation of single qubit gate

Here we describe how a single-qubit rotation can be implemented in a measurement-based way using a 3-qubit graph state. The graph state is given by

$$|G_3\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\,|+\rangle\,|+\rangle + |1\rangle\,|-\rangle\,|-\rangle\right). \tag{1}$$

The rotation angle α is realized via the choice of the measurement basis of the remaining qubit at the top (here qubit 1): it is measured in the basis $\{e^{i\alpha}\,|0\rangle + e^{-i\alpha}\,|1\rangle\,, e^{i\alpha}\,|0\rangle - e^{-i\alpha}\,|1\rangle\}$. The measurement on the top qubit is done after the read-in and thus allows to adjust for a possible Pauli byproduct operator at the read-in. Rotations around the Z-axis can be obtained by applying a Hadamard operation on the two qubits at the bottom. Since any single qubit rotation can be decomposed into X- and Z-rotations this allows for the implementation of arbitrary rotations.

If the rotation is a single qubit Clifford gate the measurement at the top qubit will be a Pauli measurement and can be done beforehand, leading to a smaller resource state.

Measurement-based implementation of two qubit gate

A measurement-based implementation of the controlled phase (CZ) gate can be done with a state

$$|G_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)/2 \tag{2}$$

The read-in is done on qubits 1 and 3, the output qubits are qubits 2 and 4. Potential Pauli byproduct operators from the read-in can be commuted through the CZ gate, leading to new Pauli operators on the output qubits. Consequently the implementation is deterministic.

EXAMPLES FOR CODES

Repetition code

Let us consider the code with codewords $|0_L\rangle = |+\rangle\,|+\rangle\,|+\rangle$ and $|1_L\rangle = |-\rangle\,|-\rangle\,|-\rangle$. It can correct a single qubit Z error on any of the three physical qubits. The resource state which allows one to do the encoding into this code is given by

$$\frac{1}{\sqrt{2}}(|0\rangle|0_L\rangle + |1\rangle|1_L\rangle) \tag{3}$$

This state is up to local Clifford operations equivalent to the graph state shown in Fig. 1a (right). Note that by choosing $|0_L\rangle = |0\rangle\,|0\rangle\,|0\rangle$ and $|1_L\rangle = |1\rangle\,|1\rangle\,|1\rangle$ one obtains a code which can protect against single qubit X errors.

The decoding can be done with the same state, where qubits 2, 3, 4 are used as input, and qubit 1 as output (see Fig. 1a, left). The results of the Bell measurements also reveal the error syndrome, i.e. we actually have decoding with built-in error correction.

Cluster-ring code

The codes described above cannot protect against an arbitrary single-qubit error. This can be achieved with the optimal five-qubit code. The codewords correspond to graph states of a closed 5-qubit ring, with $|0_L\rangle$ [$|1_L\rangle$] being the

+1 [−1] eigenstate with respect to all correlation operators $K_j = Z^{(j-1)}X^{(j)}Z^{(j+1)}$ (addition is understood modulo 5), $K_j|0_L\rangle = |0_L\rangle$, $K_j|1_L\rangle = -|1_L\rangle$ for all j and $|1_L\rangle = Z^{\otimes 5}|0_L\rangle$. It is easy to see that any single-qubit Pauli error maps the logical subspace onto an orthorgonal error-subspace. The resource state which allows to do the encoding into this code is given by

$$\frac{1}{\sqrt{2}}(|0\rangle|0_L\rangle + |1\rangle|1_L\rangle). \tag{4}$$

This state is up to local Clifford operations equivalent to the graph state shown in Fig. 1b (right).

Again, the same resource state allows one to perform the decoding with built-in error correction.

MOVING THE NOISE

In [3] we have shown that one can exchange the location of local depolarizing noise (LDN) when followed by a Bell measurement. To be precise

$$\mathcal{P}_B\mathcal{D}^1\rho = \mathcal{P}_B\mathcal{D}^2\rho, \tag{5}$$

where $\mathcal{P}_B\rho = P_B\rho P_B^\dagger$ and P_B denotes a projector on a Bell state.

This identity makes it easier to analyze the quantum information processing with noisy resource states with only input and output qubits. The reason is that one can divide the process into three steps. First, the noise on the input qubits of the resource states is moved to the input qubits which shall be processed. Second, the perfect, noise free circuit encoded into the resource state is applied. In the third and last step, the noise acting on the output qubits of the resource state is applied to the output of the circuit. The simplification comes from the fact that often analytical expressions are available for the ideal circuits (step 2), but not for the noisy ones.

CODE SWITCH

A resource state capable of code switching can be obtained by combining a resource state for decoding for one code with a resource state for encoding for a different code via a Bell measurement. The Bell measurement is done beforehand, so that the state of the qubit is never decoded and quantum information remains protected. This is illustrated in Fig. 1b in the main paper for switching between the repetition and the ring cluster code. The Bell measurement changes the structure of the resulting graph state. Note that the measurement results at the read-in provide the error syndrome, and hence the required (Pauli) correction operation at the output particles. The required correction operation can be determined by considering the two virtual steps separately. The decoding with built-in error correction leads to a Pauli correction for the output particle of this (virtual) step. This correction operation is then mapped by the encoding circuit to a Pauli operation (possibly affecting many qubits) on the output particles of the second step. Note that one does not have to implement these Pauli operations physically, it is sufficient to update the Pauli frame.

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[3] Zwerger, M., Briegel, H. J., and Dür, W. Universal and Optimal Error Thresholds for Measurement-Based Entanglement Purification. , *Phys. Rev. Lett.* **110**, 260503 (2013).